Passarino-Veltman integrals and tensor reduction

H. Hluchá

Institute for High-energy Physics
Austrian Academy of Sciences
Vienna

Svit, 2008

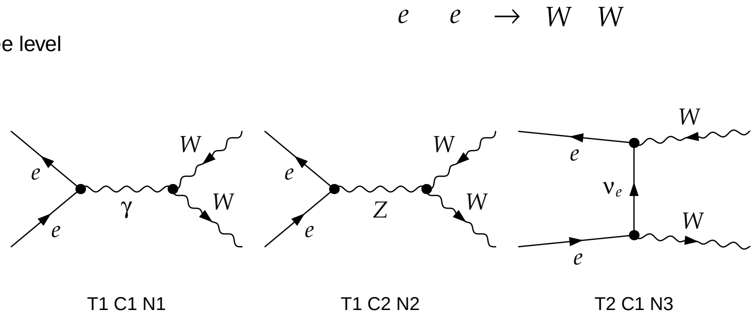
Outline

Example of a process

- Imagine a calculation of the process $e^+e^- \to W^+W^-$

  question: How many Feynman diagrams does one have to calculate?

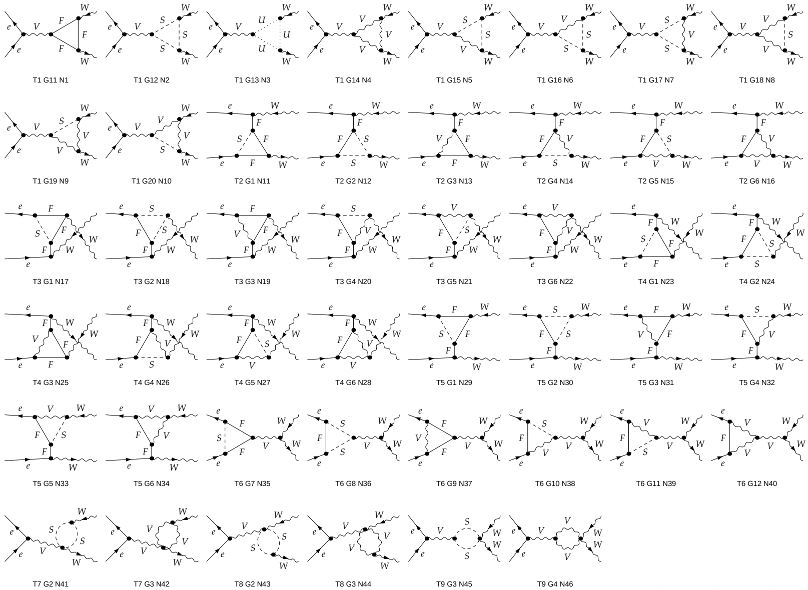
Tree level



diagrams generated by FeynArts

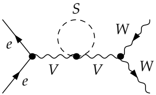
vertices

$e \quad e \quad \rightarrow \quad W \quad W$

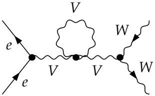


self-energies

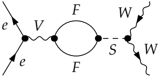
$$e \quad e \rightarrow W \quad W$$



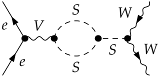
T1 G1 N1



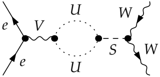
T1 G2 N2



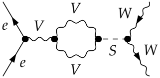
T2 G1 N3



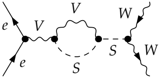
T2 G2 N4



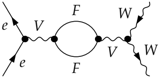
T2 G3 N5



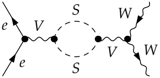
T2 G4 N6



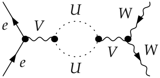
T2 G5 N7



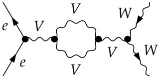
T2 G6 N8



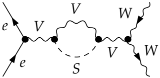
T2 G7 N9



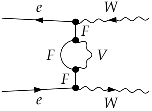
T2 G8 N10



T2 G9 N11



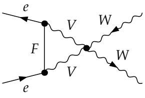
T2 G10 N12



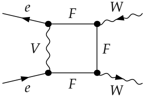
T3 G1 N13

boxes

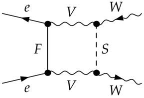
$$e \quad e \quad \rightarrow \quad W \quad W$$



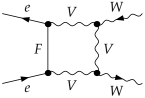
T1 G1 N1



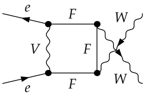
T2 G1 N2



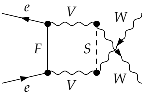
T2 G2 N3



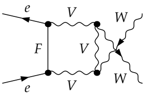
T2 G3 N4



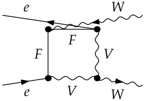
T3 G1 N5



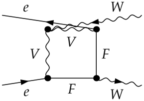
T3 G2 N6



T3 G3 N7



T4 G1 N8



T4 G2 N9

Motivation

- question: How many Feynman diagrams does one have to calculate?

  $\hookrightarrow$ the answer is: MANY

- question: How many loop integrals does one have to really calculate?

  $\hookrightarrow$ the answer is: only FEW

- Any loop integral can be decomposed into a few "scalar" integrals. The decomposition is called *tensor reduction*.

- So the message here is that all amplitudes resulting from calculations of Feynman diagrams can be written only through a few number of basic integrals.

Dimensional regularization

- **aim**: To make the divergence explicit.
- **way**: By lowering the dimension of integral. Initially divergent integrals can be thus made finite
- note: DimReg preserves gauge invariance and Poincaré invariance unlike e.g cut-off scheme.

  example:

$$\int_a^\infty \frac{1}{\vec{x}^2} d^3\vec{x} \sim x \qquad \int_a^\infty \frac{1}{\vec{x}^2} d^2\vec{x} \sim \ln x \qquad \int_a^\infty \frac{1}{x^2} dx \sim \frac{1}{x}$$

  linear divergence logarithmic divergence convergent

- Minkowski space is D-dimensional (one dimensional time together with (D-1)-dimensional Euclidean space) where D < 4. It is hard to imagine D-dimensional space, however, we can construct a formal calculus in D-dimensions.

Dimensional regularization

- notes:

  $*$ Formal calculus = set of consistent formal rules.

  $*$ The limit D $\longrightarrow$ integer number leads to ordinary integration.

  $*$ Results obtained by DimReg can be checked by other more physical methods (e.g lattice calculations). But it takes more time and more effort.

- rules for calculation in D-dimensions

  $*$ metric : $g^{\mu\nu}$ is D-dimensional, $\mu, \nu = 0, 1, \ldots D-1$
  $g^{\mu}_{\mu} = D$

  $*$ Dirac matrices: $\{\gamma^{\mu}\gamma^{\nu}\} = 2g^{\mu\nu}$
  $Tr\ 1 = 4$

- useful formulas:

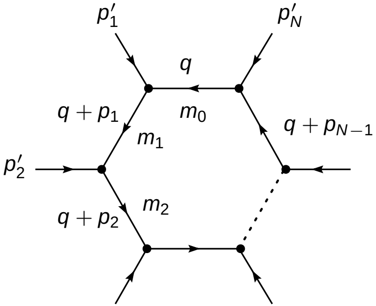
$$\gamma^{\mu}\gamma_{\mu} = \frac{1}{2}\{\gamma^{\mu}, \gamma_{\mu}\} = g^{\mu}_{\mu} = D$$

$$\gamma^{\mu}\gamma_{\nu}\gamma_{\mu} = 2g_{\mu\nu}\gamma^{\mu} - \gamma^{\mu}\gamma_{\mu}\gamma_{\nu} = (2-D)\gamma_{\nu}$$

Standard form of Pa-Ve integral

- Standard one-loop integral

$$T^N_{\mu_1\ldots\mu_M}(p_1,\ldots,p_{N-1},m_0,\ldots,m_{N-1}) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^Dq \frac{q_{\mu_1}\ldots q_{\mu_M}}{[q^2-m_0^2+i\varepsilon][(q+p_1)^2-m_1^2+i\varepsilon]\ldots[(q+p_{N-1})^2-m_{N-1}^2+i\varepsilon]}$$



$$p_1 = p'_1$$
$$p_2 = p'_2 + p'_1$$
$$\vdots$$
$$p_{N-1} = p'_{N-1} + p'_{N-2}$$

Decomposition

- Decomposition: allowed due to Lorentz covariance in D-dimensions

$$\begin{aligned} B^{\mu} &= p_1^{\mu} B_1 \\ B^{\mu\nu} &= g^{\mu\nu} B_{00} + p_1^{\mu} p_1^{\nu} B_{11} \\ C^{\mu} &= p_1^{\mu} C_1 + p_2^{\mu} C_2 \\ C^{\mu\nu} &= g^{\mu\nu} C_{00} + p_1^{\mu} p_1^{\nu} C_{11} + (p_1^{\mu} p_2^{\nu} + p_2^{\mu} p_1^{\nu}) C_{12} + p_2^{\mu} p_2^{\nu} C_{22} \\ &\vdots \end{aligned}$$

- All loop integrals can be reduced to basic "scalar" integrals A_0, B_0, C_0, D_0, $\ldots$. They do not contain any Lorentz index in the numerator.

  So it means that integrals $A^{\mu\nu}, A_2, B^{\mu}, B_1, \ldots$ can be expressed only through scalar integrals.

Generic integral

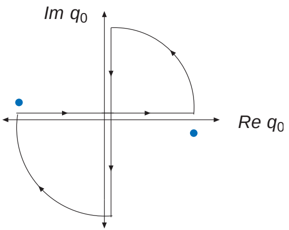
- The basic integral with the help of which Pa-Ve integrals can be computed is of the form

$$I_n(A) = \int d^D q \frac{1}{(q^2 - A + i\varepsilon)^n}$$

It is convergent providing the dimension $D < 2n$ and $A > 0$.

- Calculation of the generic integral

1. Poles of the integrand: $q_0 = \pm\sqrt{\vec{q}^2 + A} \mp i\varepsilon'$
2. Wick rotation:



Cauchy theorem: $\oint_C (\ldots) = 0$

arcs give zero

$\int_{-\infty}^{\infty} dq_0 \ldots = \int_{-i\infty}^{i\infty} dq_0 \ldots$

$I_n(A) = \int_{-i\infty}^{i\infty} dq_0 \int d^{D-1}q (q^2 - A + i\varepsilon)^{-n}$

Generic integral

3. Substitution: $q_0 = iq_{E,0}, \quad q^k = q_E^k, \quad q^2 = -q_E^2$

   after which the integral has the form

$$I_n(A) = i \int d^D q_E (-1)^n (q_E^2 + A - i\varepsilon)^{-n}$$

4. Solid angle Ω_D in D dimensions

$$\begin{aligned}(\sqrt{\pi})^D &= \left(\int_0^\infty dx\ e^{-x^2}\right)^D = \int_0^\infty dx^1 \ldots dx^D e^{-\sum_{i=1}^D x_i^2} \\ &= \int d\Omega_D \int_0^\infty dx \ \ x^{D-1} e^{-x^2} = \frac{1}{2}\Omega_D \Gamma(D/2)\end{aligned}$$

5. Polar coordinates

$$\int d^D q_E = \int d\Omega_D \int_0^\infty dq_E q_E^{D-1} = \int d\Omega_D \int_0^\infty dq_E^2 \frac{1}{2}(q_E^2)^{D/2-1}$$

Generic integral

- After going to spherical coordinates the generic integral becomes

$$I_n(A) = i(-1)^n \frac{\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty dx\ x^{D/2-1}(x + A - i\varepsilon)^{-n}$$

- We have got a simple one-dimensional integral. It can be solved by using Beta and Gamma functions.

  with the substitution
$$y = \frac{A - i\varepsilon}{x + A - i\varepsilon}, \quad dy = -\frac{A - i\varepsilon}{(x + A - i\varepsilon^2)}$$

  we get

$$I_n(A) \quad = \quad i(-1)^n \frac{\pi^{D/2}}{\Gamma(D/2)} (A - i\varepsilon)^{D/2-n} \int_0^1 dy\ (1-y)^{(D/2-1)} y^{(n-D/2-1)}$$

  Applying identities for Beta and Gamma functions we finally come to the result

$$I_n(A) = i(-1)^n \pi^{D/2} \frac{\Gamma(n - D/2)}{\Gamma(n)} (A - i\varepsilon)^{D/2-n}$$

Scalar integral A_0

- The definition is:

$$A_0(m^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \; (q^2 - m^2 + i\varepsilon)^{-1}$$

$\hookrightarrow$ parameter μ has dimension of energy and serves for retaining the dimension of the integral

- With the help of the generic integral we can write

$$A_0(m^2) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} I_1(m^2) = -m^2 \left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{D-4}{2}} \Gamma\left(\frac{2-D}{2}\right)$$

- Integral is divergent sending $D \to 4$ b/c $\Gamma(z)$ has pole at z = -1.
- Introducing $\varepsilon = (4-D)/2$ integral A_0 gets the form

$$A_0(m^2) = -m^2 \left(\frac{m^2}{4\pi\mu^2}\right)^{-\varepsilon} \Gamma(\varepsilon - 1)$$

Scalar integral A_0

- Next step is to have A_0 in ε-series expansion

$$\left(\frac{m^2}{4\pi\mu^2}\right)^{-\varepsilon} = \exp\left[-\varepsilon \ln\left(\frac{m^2}{4\pi\mu^2}\right)\right] = 1 - \varepsilon \ln\left(\frac{m^2}{4\pi\mu^2}\right) + \mathcal{O}(\varepsilon^2)$$

$$\Gamma(\varepsilon - 1) = \frac{1}{\varepsilon(\varepsilon-1)}\Gamma(1+\varepsilon) = -\frac{1}{\varepsilon} - \Gamma(1) + \gamma_E + \mathcal{O}(\varepsilon)$$

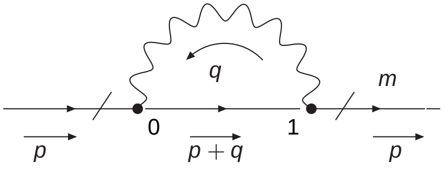
where γ_E is the Euler-Mascharoni constant $\gamma_E = -\Gamma'(1) = 0.5772$

- Collecting the terms up to the first power of ε the scalar integral A_0 finally is

$$\begin{aligned} A_0(m^2) &= m^2\left(\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi - \ln\left(\frac{m^2}{\mu^2}\right) + 1 + \mathcal{O}(\varepsilon)\right) \\ &= m^2\left(\Delta - \ln\left(\frac{m^2}{\mu^2}\right) + 1 + \mathcal{O}(\varepsilon)\right) \end{aligned}$$

with $\Delta = \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi$

Electron self-energy



$0: \quad ie\gamma^{\mu}$

$1: \quad ie\gamma^{\nu}$

$B(p^2, 0, m^2)$

- Calculation of the self-energy

$$
\begin{aligned}
-i\,\Sigma &= \mu^{4-D}\int \frac{d^Dq}{(2\pi)^D}\,(i\gamma^{\nu}e)\,\frac{-ig_{\mu\nu}}{q^2+i\varepsilon}\,\frac{i(\not{p}+\not{q}+m)}{(q+p)^2-m^2+i\varepsilon}\,(i\gamma^{\mu}e)\\
&= \frac{ie^2}{(4\pi)^2}\,\frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq\,(-1)\gamma_{\mu}\frac{(\not{p}+\not{q}+m)}{\mathcal{D}_0\mathcal{D}_1}\gamma^{\mu}\\
&\qquad \gamma_{\mu}\gamma^{\mu} = D, \qquad\qquad \gamma^{\mu}\gamma_{\nu}\gamma_{\mu} = (2-D)\gamma_{\nu}\\
&= \frac{ie^2}{(4\pi)^2}\int_q \frac{(D-2)(\not{p}+\not{q})-Dm}{\mathcal{D}_0\mathcal{D}_1}\\
&= \frac{ie^2}{(4\pi)^2}\left[(D-2)\not{p}B_0 - DmB_0 + (D-2)\gamma^{\mu}B_{\mu}\right]
\end{aligned}
$$

Reduction of the B^μ integral

- Decomposition: $B^\mu = p^\mu B_1$
- We contract the equation by p_μ (which is all we have at our disposal)

$$p^2 B_1 = p^\mu B_\mu = \int_q d^D q \frac{p.q}{(q^2+i\varepsilon)[(q+p)^2-m^2+i\varepsilon]}$$

$$p.q = \frac{1}{2}\Big([(q+p)^2-m^2+i\varepsilon]-(q^2+i\varepsilon)-(p^2-m^2)\Big)$$

$$= \frac{1}{2}\Big(\int_q \frac{1}{q^2+i\varepsilon} - \int_q \frac{1}{(q+p)^2-m^2+i\varepsilon}$$

$$- (p^2-m^2)\int_q \frac{1}{(q^2+i\varepsilon)[(q+p)^2-m^2+i\varepsilon]}\Big)$$

- Solving this simple equation for B_1 we get

$$B_1(p^2,0,m^2) = \frac{1}{2p^2}[A_0(0)-A_0(m^2)-(p^2-m^2)B_0(p^2,0,m^2)]$$

Electron self-energy (continued)

- Let us come back to the calculation of the amplitude

$$-i\,\Sigma = \frac{ie^2}{(4\pi)^2}[(D-2)\not{p}B_0 - DmB_0 + (D-2)\gamma^\mu B_\mu]$$
$$= \frac{ie^2}{(4\pi)^2}[(D-2)\not{p}(B_0+B_1) - DmB_0]$$

$\hookrightarrow$ We still cannot send $D \to 4$ but have to do proper limit of DB_0 and DB_1 first. For this purpose I show the following table.

- UV divergent part of some Pa-Ve integrals

Integral		UV divergent part
$A_0(m^2)$	$\longrightarrow$	$m^2\Delta$
B_0	$\longrightarrow$	Δ
B_1	$\longrightarrow$	$-\frac{1}{2}\Delta$
$B_{00}(k^2, m_0^2, m_1^2)$	$\longrightarrow$	$-\frac{1}{4}(k^2/3 - m_0^2 - m_1^2)\Delta$
B_{11}	$\longrightarrow$	$\frac{1}{3}\Delta$
C_{00}	$\longrightarrow$	$\frac{1}{4}\Delta$

Electron self-energy (continued)

- Proper limit of DB_0, DB_1:

$$
\begin{aligned}
DB_0 &= (4-2\varepsilon)\left(\frac{1}{\varepsilon}-\gamma_E+\ln 4\pi+\dots\right)\to 4B_0-2\\
DB_1 &= (4-2\varepsilon)\left[-\frac{1}{2}\left(\frac{1}{\varepsilon}-\gamma_E+\ln 4\pi\right)+\dots\right]\to 4B_1+1
\end{aligned}
$$

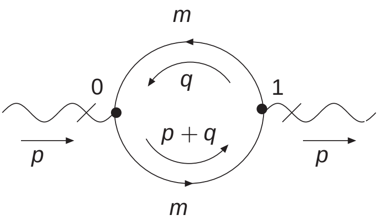
- So the electron SE finally is:

$$
\begin{aligned}
-i\,\Sigma &= \frac{ie^2}{(4\pi)^2}[(D-2)\not{p}(B_0+B_1)-DmB_0]\\
&= \frac{ie^2}{(4\pi)^2}[(2B_0+2B_1-1)\not{p}-m(4B_0-2)]
\end{aligned}
$$

where the argument of all B integrals is $B(p^2, 0, m^2)$

- The whole $-i\,\Sigma$ can be expressed through A_0 and B_0 integrals only.

Virtual photon self-energy



$0: \quad ig_0\gamma^\mu$

$1: \quad ig_1\gamma^\nu$

$B(p^2, m^2, m^2)$

$g_i = e$ for the electron

$$-i\,\Sigma \quad = \quad \frac{i}{(4\pi)^2}\int_q (-1)\frac{1}{\mathcal{D}_1\mathcal{D}_2} Tr[\gamma^\mu g_0(\not{q}+m)\gamma^\nu g_1(\not{p}+\not{q}+m)]$$

traces:

$$Tr(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$
$$Tr(\gamma^\mu\gamma^\nu\gamma^\rho) = 0$$
$$Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$

$$-i\,\Sigma = \frac{-4ig_0g_1}{(4\pi^2)}\int_q \frac{1}{\mathcal{D}_1\mathcal{D}_2}\left\{q^\mu(p+q)^\nu - g^{\mu\nu}\left[q.(p+q)\right] + q^\nu(p+q)^\mu + m^2g^{\mu\nu}\right\}$$

Reduction of $B^{\mu\nu}$

- Decomposition: $$B^{\mu\nu} = g^{\mu\nu} B_{00} + p^\mu p^\nu B_{11}$$
- Contracting with metric tensor and with momentum we get the system

$$\begin{aligned} g_{\mu\nu} B^{\mu\nu} &= D B_{00} + p^2 B_{11} \\ p_\mu B^{\mu\nu} &= p^\nu (B_{00} + p^2 B_{11}) \end{aligned}$$

- Let us now calculate the LHSs in order to have them in terms of A_0, B_0 integrals

$$\begin{aligned} g_{\mu\nu} B^{\mu\nu} &= \int_q d^D q \frac{q^2}{(q^2 - m_0^2 + i\varepsilon)[(q+p)^2 - m_1^2 + i\varepsilon]} \\ &= A_0(m_1^2) + m_0^2 B_0(p^2, m_0^2, m_1^2) \end{aligned}$$

$$p_\mu B^{\mu\nu} = \int_q d^D q \frac{(p.q) q^\nu}{(q^2 - m_0^2 + i\varepsilon)[(q+p)^2 - m_1^2 + i\varepsilon]}$$

- Now use:

$$(p.q) = \frac{1}{2} \Big([(q+p)^2 - m^2 + i\varepsilon] - (q^2 - m_0^2 + i\varepsilon) - (p^2 - m^2 - m_0^2) \Big)$$

Reduction of $B^{\mu\nu}$

- To get

$$p_\mu B^{\mu\nu} = \int_q \frac{1}{2}\Bigg(\frac{q^\nu}{q^2 - m_0^2 + i\varepsilon} - \frac{q^\nu}{(q+p)^2 - m_1^2 + i\varepsilon}$$
$$-(p^2 - m_1^2 + m_0^2)\frac{q^\nu}{(q^2 - m_0^2 + i\varepsilon)[(q+p)^2 - m_1^2 + i\varepsilon]}\Bigg)$$

- From D-dimensional calculus we need

$$\int d^D q \frac{q^\nu}{(q^2 - m_0^2 + i\varepsilon)} = 0$$

- The second LHS then becomes

$$p_\mu B^{\mu\nu} = \frac{1}{2} p^\nu \left[A_0(m_1^2) - (p^2 - m_1^2 + m_0^2) B_1(p^2, m_0^2, m_1^2)\right]$$

- Solving the system we would obtain B_{00} and B_{11} expressed through integrals A_0, B_0, B_1 and still with dimension D in the expression. Last step is doing the proper limit $D \to 4$ like before.

Photon self-energy (continued)

- The amplitude after the reduction becomes

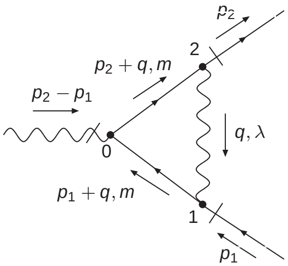
$$
\begin{aligned}
-i\,\Sigma &= \frac{-4ig_0g_1}{(4\pi^2)}\int_q \frac{1}{\mathcal{D}_1\mathcal{D}_2}\left\{q^\mu(p+q)^\nu - g^{\mu\nu}\left[q.(p+q)\right] + q^\nu(p+q)^\mu + m^2 g^{\mu\nu}\right\} \\
&= \frac{-4ig_0g_1}{(4\pi)^2}\left\{g^{\mu\nu}\left[m^2B_0 + (2-D)B_{00} - p^2B_{11} - p^2B_1\right]\right. \\
&\quad \left. +p^\mu p^\nu\left[2B_1 + 2B_{11}\right]\right\}
\end{aligned}
$$

- And after some cosmetic changes the final result is

$$
\begin{aligned}
-i\,\Sigma &= \frac{-4ig_0g_1}{(4\pi)^2}\left\{g^{\mu\nu}\left[m^2B_0 - (p^2B_1 + m^2B_0 + A_0 - 2B_{00})\right]\right. \\
&\quad \left. +p^\mu p^\nu\left[2B_1 + 2B_{11}\right]\right\}\Big\}
\end{aligned}
$$

- This self-energy can be also expressed only through A_0, B_0 integrals.

QED vertex



$0: \quad ie\gamma^\mu$

$1: \quad ie\gamma^\rho$

$2: \quad ie\gamma^\sigma$

$C(p_1^2, (p_2 - p_1)^2, p_2^2, \lambda^2, m^2, m^2)$

- Calculation of the vertex correction Λ^μ

$$\begin{aligned}\Lambda^\mu &= \mu^{4-D} \int \frac{d^D q}{(2\pi)^D} (ie\gamma^\sigma) \frac{i(\not{p}_2 + \not{q} + m)}{[(q+p_2)^2 - m^2 + i\varepsilon]} (ie\gamma^\mu) \\ &\qquad \frac{i(\not{p}_1 + \not{q} + m)}{[(q+p_1)^2 - m^2 + i\varepsilon]} (ie\gamma^\rho) \frac{-ig_{\rho\sigma}}{q^2 - \lambda^2 + i\varepsilon} \\ &= \frac{ie^3}{(4\pi)^2} \int_q \frac{1}{\mathcal{D}_0 \mathcal{D}_1 \mathcal{D}_2} \gamma_\rho (\not{q} + \not{p}_2 + m) \gamma^\mu (\not{q} + \not{p}_1 + m) \gamma^\rho\end{aligned}$$

QED vertex

- Identities:

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma_\mu &= (2-D)\gamma^\nu \\ \gamma^\rho\gamma^\mu\gamma^\nu\gamma_\rho &= 4g^{\mu\nu} + (D-4)\gamma^\mu\gamma^\nu \\ \gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\nu\gamma_\mu &= -2\gamma^\nu\gamma^\sigma\gamma^\rho + (4-D)\gamma^\rho\gamma^\sigma\gamma_\nu\end{aligned}$$

- Our vertex correction Λ^μ

$$\begin{aligned}\Lambda^\mu &= \frac{ie^3}{(4\pi)^2}\int_q \frac{1}{\mathcal{D}_0\mathcal{D}_1\mathcal{D}_2}\Big\{(2-D)\not{q}\gamma^\mu\not{q} - 2[\not{p}_1\gamma^\mu\not{q} + \not{q}\gamma^\mu\not{p}_2 + \not{p}_1\gamma^\mu\not{p}_2] \\ &+ (4-D)[\not{q}\gamma^\mu\not{p}_1 + \not{p}_2\gamma^\mu\not{q} + \not{p}_2\gamma^\mu\not{p}_1] + (D-4)m[\not{q}\gamma^\mu + \gamma^\mu\not{q} + \not{p}_2\gamma^\mu + \gamma^\mu\not{p}_1] \\ &+ (2-D)m^2\gamma^\mu + 4m(2q+p_1+p_2)^\mu\Big\}\end{aligned}$$

- Reduction: $C^\mu = p_1^\mu C_1 + p_2^\mu C_2$

$$C^{\mu\nu} = g^{\mu\nu}C_{00} + p_1^\mu p_1^\nu C_{11} + (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu)C_{12} + p_2^\mu p_2^\nu C_{22}$$

Reduction of C^μ

- Decomposition $$C^\mu = p_1^\mu C_1 + p_2^\mu C_2$$
- By contracting with p_1^μ and p_2^μ we get the system

$$\begin{aligned}(p_1)_\mu C^\mu &= p_1^2 C_1 + (p_1.p_2) C_2 \\ (p_2)_\mu C^\mu &= (p_1.p_2) C_1 + p_2^2 C_2\end{aligned}$$

- Both LHSs can be written in terms of scalar integrals (B_0, C_0). Having done this one can solve the above system of equations for C_1, C_2 providing the corresponding matrix is invertible.
- note: not invertible at the threshold when $p_1^2 p_2^2 = (p_1.p_2)^2$. (almost no velocity of colliding particles or decay to particles with almost no velocity)
- Let us first rewrite the LHSs in terms of scalar integrals

$$(p_1)_\mu C^\mu = \int_q d^D q \frac{p_1.q}{(q^2 - m_0^2 + i\varepsilon)[(q+p_1)^2 - m_1^2 + i\varepsilon][(q+p_2)^2 - m_2^2 + i\varepsilon]}$$

Reduction of C^μ

$$
\begin{aligned}
&= \int_q \frac{\frac{1}{2}([(q+p_1)^2 - m_1^2 + i\varepsilon] - (q^2 - m_0^2 + i\varepsilon) - (p_1^2 - m_1^2 + m_0^2))}{(q^2 - m_0^2 + i\varepsilon)[(q+p_1)^2 - m_1^2 + i\varepsilon][(q+p_2)^2 - m_2^2 + i\varepsilon]} \\
&= \frac{1}{2}B_0(p_2^2, m_0^2, m_2^2) - \frac{1}{2}B_0((p_1-p_2)^2, m_1^2, m_2^2) - \frac{1}{2}f_1 C_0(p_1^2, (p_1-p_2)^2, p_2^2, m_0^2, m_1^2, m_2^2) \\
&= UV \ finite
\end{aligned}
$$

with $f_i = p_i^2 - m_i^2 + m_0^2$

$$
\begin{aligned}
(p_2)_\mu C^\mu &= \int_q \frac{p_2.q}{(q^2 - m_0^2 + i\varepsilon)[(q+p_1)^2 - m_1^2 + i\varepsilon][(q+p_2)^2 - m_2^2 + i\varepsilon]} \\
&= \frac{1}{2}B_0(p_1^2, m_0^2, m_1^2) - \frac{1}{2}B_0((p_1-p_2)^2, m_1^2, m_2^2) - \frac{1}{2}f_2 C_0(p_1^2, (p_1-p_2)^2, p_2^2, m_0^2, m_1^2, m_2^2)
\end{aligned}
$$

- Now it only remains to solve the system and get the integrals C_1, C_2. We will not do it here. These integrals would be written in terms of the scalar integrals A_0, B_0, and C_0.

Reduction of $C^{\mu\nu}$

- Decomposition

$$C^{\mu\nu} = g^{\mu\nu}C_{00} + p_1^\mu p_1^\nu C_{11} + (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu)C_{12} + p_2^\mu p_2^\nu C_{22}$$

- By contracting with p_1^μ, p_2^μ and $g_{\mu\nu}$ we get

$$\begin{aligned}
(p_1)_\mu C^{\mu\nu} &= p_1^\nu C_{00} + p_1^\nu p_1^2 C_{11} + p_2^\nu p_1^2 C_{12} + p_1^\nu (p_1.p_2) C_{12} + p_2^\nu (p_1.p_2) C_{22} \\
(p_2)_\mu C^{\mu\nu} &= p_2^\nu C_{00} + p_1^\nu (p_1.p_2) C_{11} + p_1^\nu p_2^2 C_{12} + p_2^\nu (p_1.p_2) C_{12} + p_2^\nu p_2^2 C_{22} \\
g_{\mu\nu} C^{\mu\nu} &= D C_{00} + p_1^2 C_{11} + 2 p_1.p_2 C_{12} + p_2^2 C_{22}
\end{aligned}$$

5 equations for 4 variables $C_{00}, C_{11}, C_{12}, C_{22}$

- Rewriting the LHSs we get

$$g_{\mu\nu}C^{\mu\nu} = \int_q \frac{q^2}{\mathcal{D}_0\mathcal{D}_1\mathcal{D}_2} = \int_q \frac{q^2 - m_0^2 + m_0^2}{\mathcal{D}_0\mathcal{D}_1\mathcal{D}_2} = B_0((p_2 - p_1)^2, m_1^2, m_2^2) + m_0^2 C_0$$

Decomposition of $C^{\mu\nu}$

$$
\begin{aligned}
(p_1)_\mu C^{\mu\nu} &= \int_q \frac{(p_1.q)q^\nu}{\mathcal{D}_0\mathcal{D}_1\mathcal{D}_2} = \int_q \frac{q^\nu \frac{1}{2}(\mathcal{D}_1 - \mathcal{D}_0 - f_1)}{\mathcal{D}_0\mathcal{D}_1\mathcal{D}_2} \\
&= \int_q \frac{1}{2}\left(\frac{q^\nu}{\mathcal{D}_0\mathcal{D}_2} - \frac{q^\nu}{\mathcal{D}_1\mathcal{D}_2} - f_1\frac{q^\nu}{\mathcal{D}_0\mathcal{D}_1\mathcal{D}_2}\right) \\
&= \frac{1}{2}p_2^\nu B_1(p_2^2, m_0^2, m_2^2) + \frac{1}{2}(p_1 - p_2)^\nu B_1((p_2 - p_1)^2, m_1^2, m_2^2) \\
&+ \frac{1}{2}p_1^\nu B_0((p_2 - p_1)^2, m_1^2, m_2^2) - \frac{1}{2}p_1^\nu f_1 C_1 - \frac{1}{2}p_2^\nu f_1 C_2
\end{aligned}
$$

- analogously $(p_2)_\mu C^{\mu\nu}$

$$
\begin{aligned}
(p_2)_\mu C^{\mu\nu} &= \frac{1}{2}p_1^\nu B_1(p_1^2, m_0^2, m_1^2) + \frac{1}{2}(p_1 - p_2)^\nu B_1((p_2 - p_1)^2, m_1^2, m_2^2) \\
&+ \frac{1}{2}p_1^\nu B_0((p_2 - p_1)^2, m_1^2, m_2^2) - \frac{1}{2}p_1^\nu f_2 C_1 - \frac{1}{2}p_2^\nu f_2 C_2
\end{aligned}
$$

Reduction of $C^{\mu\nu}$

- 5 equations now have the following form

$$
\begin{aligned}
DC_{00} + p_1^2 C_{11} + 2p_1.p_2 C_{12} + p_2^2 C_{22} &= B_0((p_2 - p_1)^2, m_1^2, m_2^2) + m_0^2 C_0 && (1)\\
C_{00} + p_1^2 C_{11} + p_1.p_2 C_{12} &= \frac{1}{2}[B_0((p_2 - p_1)^2, m_1^2, m_2^2) + \\
&\quad + B_1((p_2 - p_1)^2, m_1^2, m_2^2) - f_1 C_1] && (2)\\
p_1^2 C_{12} + p_1.p_2 C_{22} &= \frac{1}{2}[B_1(p_2^2, m_0^2, m_2^2) - \\
&\quad - B_1((p_2 - p_1)^2, m_1^2, m_2^2) - f_1 C_2] && (3)\\
p_1.p_2 C_{11} + p_2^2 C_{12} &= \frac{1}{2}[B_1(p_1^2, m_0^2, m_1^2) + B_1((p_2 - p_1)^2, m_1^2, m_2^2) + \\
&\quad + B_0((p_2 - p_1)^2, m_1^2, m_2^2) - f_2 C_1] && (4)\\
C_{00} + p_1.p_2 C_{12} + p_2^2 C_{22} &= \frac{1}{2}[-B_1((p_2 - p_1)^2, m_1^2, m_2^2) - f_2 C_2] && (5)
\end{aligned}
$$

- From eq. (1) we obtain C_{00} which we can substitute to four remaining equations. From eqs. (2) and (4) we obtain C_{11}, C_{12} and from eqs. (3) and (5) we obtain C_{12}, C_{22}.

Reduction of B^μ and the threshold

- Recall: $B_1(p^2, 0, m^2) = \frac{1}{2p^2}[A_0(0) - A_0(m^2) - (p^2 - m^2)B_0(p^2, 0, m^2)]$
- Not allowed when $p^2 = 0$ (e.g photon self-energy)
- Where is the problem?
  $\hookrightarrow$ The problem lies in the decomposition. B_μ must be calculated directly.

$$B^\mu = \int_q \frac{q^\mu}{(q^2 - m_0^2 + i\varepsilon)[(q + p_1)^2 - m_1^2 + i\varepsilon]}$$

- First step: Feynman parametrization: $\frac{1}{ab} = \int_0^1 dx \frac{1}{[a(1-x)+bx]^2}$

$$a = q^2 - m_0^2 + i\varepsilon, \qquad b = (q + p_1)^2 - m_1^2 + i\varepsilon$$

$$B^\mu = \int_0^1 dx \int_q q^\mu \{(q + xp)^2 - x^2p^2 + x(p^2 - m_1^2 + m_0^2) - m_0^2 + i\varepsilon\}$$

- Second step: substitution

$$\begin{aligned} q' &= q + xp, \qquad dq' = dq \\ A &= x^2p^2 - x(p^2 - m_1^2 + m_0^2) + m_0^2 \end{aligned}$$

Reduction of B^μ and the threshold

- After the substitution

$$B^\mu = \int_0^1 \int_{q'} \frac{(q'-xp)^\mu}{(q'^2-A+i\varepsilon)^2} = \int_0^1 dx \int_{q'} \frac{-xp^\mu}{(q'^2-A+i\varepsilon)^2}$$

- This is to compare with: $B^\mu = p^\mu B_1$

$$B_1 = -\int_0^1 dx \int_{q'} \frac{x}{(q'^2-A+i\varepsilon)^2} = -\int_0^1 dx\ x\ I_2(A)\frac{(2\pi\mu)^{4-D}}{i\pi^2}$$

- Using the formula for the generic integral I_n we get

$$\begin{aligned} B_1 &= -(4\pi\mu)^{\frac{4-D}{2}}\Gamma(\varepsilon)\int_0^1 dx\ x\ A^{-\varepsilon} = -\int_0^1 dx\ x\ \left\{\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi - \ln\left(\frac{A-i\varepsilon}{\mu^2}\right)\right\} \\ &= -\frac{1}{2}\Delta + \int_0^1 dx\ x\ \ln\left(\frac{x^2p^2 - x(p^2-m_1^2+m_0^2)+m_0^2-i\varepsilon}{\mu^2}\right) \end{aligned}$$

- Having the correct coefficient going with the divergent part Δ. Note that now one only have to calculate ordinary 1-dimensional integral

Summary

- Dimensional regularization:

  To make the divergence of the loop integrals explicit one can use procedure called Dimensional regularization. One lowers the dimension of the space through which one integrates and thus makes the originally divergent integrals finite. After renormalizing the theory all the divergent parts $1/\varepsilon$ of all integrals cancel out.

- Tensor reduction

  One is not forced to calculate all the scalar as well as tensor integrals which rise in the calculation of various Feynman diagrams. One only needs to know the result for the scalar integrals $A_0, B_0, C_0, \ldots$. Other integrals can be expressed through these basic integrals. Needless to say, this saves lots of CPU time.

Thank you for the attention.

Bibliography

[1] K. Kovařík: *private notes*, 2008

[2] M. Steinhauser: *Übungen zu Strahlungskorrekturen in Eichtheorien*, Herbstschule für Hochenenergiephysik, Maria Laach 2003

[3] M. Böhm, A. Denner, H. Joos: *Gauge Theories of the Strong and Electroweak Interaction*, B.G. Teubner, 2001

[4] J. Collins: *Renormalization*, Cambridge Univ. Press, 1984